

C^* -ALGEBRAS WHOSE EVERY C^* -SUBALGEBRA IS AF

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ABSTRACT. Let A be a C^* -algebra. It is shown that the following conditions are equivalent:

- (1) A is scattered,
- (2) every C^* -subalgebra of A is AF,
- (3) every C^* -subalgebra of A has real rank zero.

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1. INTRODUCTION

An AF C^* -algebra was introduced by Bratteli, which is defined as the inductive limit of a sequence of finite-dimensional C^* -algebras. As is well-known, every hereditary C^* -subalgebra of an AF C^* -algebra is also AF. But in general, an AF C^* -algebra contains C^* -subalgebras which are not AF. For example, the C^* -algebra $C(2^\omega)$ of all continuous functions on the Cantor set 2^ω is AF because the Cantor set 2^ω is a totally disconnected topological space. But $C(2^\omega)$ contains a C^* -subalgebra which is isomorphic to the C^* -algebra $C(I)$ of all continuous functions on the closed interval $I = [0, 1]$, and $C(I)$ is not AF. In §2, by an AF C^* -algebra we will mean a C^* -algebra A in which any finite set of elements can be approximated arbitrarily closely in norm by elements of a finite-dimensional C^* -subalgebra of A . This definition has a slightly wider sense than the usual one introduced by Bratteli, but an AF C^* -algebra in our sense is recently often adopted as the definition of an AF C^* -algebra. Then any scattered C^* -algebra is AF in our sense above ([10, Lemma 5.1]). However, every AF C^* -algebra is not scattered. See [9, Theorem 2.2] for conditions for AF C^* -algebras to become scattered. The purpose of this paper is to characterize C^* -algebras whose every C^* -subalgebra is AF. In fact, in §2, we will show that a C^* -algebra A is scattered if and only if every C^* -subalgebra of A is AF.

Here recall that a topological space is called *scattered* (or *dispersed*) if every non-empty subset necessarily contains an isolated point. For a compact Hausdorff space Ω , it is shown in [12] that Ω is scattered if and only if every Radon measure on Ω is atomic. As a non-commutative generalization of a scattered compact Hausdorff space, the notion of a scattered C^* -algebra was introduced independently by Jensen [7] and Rothwell [13]. We say that a C^* -algebra A is *scattered* if every positive linear functional on A is the countable sum of pure positive linear functionals on A , or equivalently, A is of type I and the spectrum \hat{A} of A is a scattered topological space equipped with the Jacobson topology ([8, Corollary 3], [13, Theorem 3.8]). The reader is referred to [3], [6], [7], [8], [9], [13] for other equivalent conditions on scattered C^* -algebras.

By the way, AF C^* -algebras of Bratteli's sense have real rank zero which was defined by Brown-Pedersen [5], and there are C^* -algebras with real rank zero which are not AF. In §2, we will show also that every C^* -subalgebra of A is AF in our sense if and only if every C^* -subalgebra of A has real rank zero. Thus we will see that scattered C^* -algebras can be completely characterized by the condition that every C^* -subalgebra should have the same property to be AF or to have real rank zero. This leads us to the question of what kind of C^* -algebra can be characterized by some property the C^* -subalgebras possess. In §3, we will show that the class of type I C^* -algebras is one of answers for such a question. In fact, we show that a C^* -algebra A is of type I if and only if every C^* -subalgebra of A is strongly amenable, if and only if every C^* -subalgebra of A is nuclear.

2. APPROXIMATELY FINITE DIMENSIONALITY AND REAL RANK ZERO

In this section, the term AF C^* -algebra has a slightly wider sense than the usual one which is assumed to be separable and to have identity. We will ease those restrictions. In fact, we say that a C^* -algebra A is an *approximately finite-dimensional* C^* -algebra (or simply *AF*) if for any elements $x_1, x_2, \dots, x_n \in A$ and any $\epsilon > 0$, there exists a finite dimensional C^* -subalgebra B such that $\|x_i - b_i\| < \epsilon$

with some $y_k \in B$ for all $k = 1, 2, \dots, n$. In this definition, any scattered C^* -algebra is AF ([10, Lemma 5.1]).

For a compact Hausdorff space Ω , we denote by $C(\Omega)$ the C^* -algebra of all continuous functions on Ω . Throughout this section, we denote by 2^ω the Cantor set. Note that the Cantor set 2^ω is totally disconnected and is not scattered. Hence $C(2^\omega)$ is AF, but it is not a scattered C^* -algebra.

Here we recall that a topological space X is called *0-dimensional* if each point of X has a neighborhood base consisting of open-closed sets. Equivalently, X is 0-dimensional if and only if for each point $x \in X$ and closed set F not containing x , there is an open and closed set containing x and not meeting F . It is well-known that a locally compact Hausdorff space is 0-dimensional if and only if it is totally disconnected.

For a C^* -algebra A , we always denote by A_1 the C^* -algebra obtained from A by adjunction of an identity. We say that a C^* -algebra A has *real rank zero* if and only if every self-adjoint element in A_1 can be approximated by an invertible self-adjoint element in A_1 , or equivalently, A has the (FS) property, that is, the set of all self-adjoint elements with finite spectrum of A is dense in the set of all self-adjoint elements of A ([5, Theorem 2.6]).

Lemma 2.1. *Let Ω be a compact Hausdorff space and let $C(\Omega)$ be the C^* -algebra of all continuous functions on Ω . If every C^* -subalgebra of $C(\Omega)$ has real rank zero, then $C(\Omega)$ is scattered.*

Proof. Assume that $C(\Omega)$ is not a scattered C^* -algebra. Then Ω is not scattered as a topological space (see [8, Corollary 3]). On the other hand, since $C(\Omega)$ has real rank zero by assumption, it follows from [5, 1.1] that Ω is 0-dimensional. Hence it follows from [12, 2. Main Theorem] that there exists a continuous map π from Ω onto the Cantor set 2^ω . Hence we obtain that $C(2^\omega)$ can be embedded into $C(\Omega)$ as a C^* -algebra via the induced map $\tilde{\pi}$ defined by $\tilde{\pi}(f) = f \circ \pi$ for $f \in C(2^\omega)$. Since there is a continuous map from 2^ω onto the closed interval $I = [0, 1]$, $C(I)$ can be embedded into $C(2^\omega)$ as a C^* -algebra. Hence $C(I)$ can be embedded into $C(\Omega)$ as a C^* -algebra. Since I is not totally disconnected, it is not 0-dimensional. Hence $C(I)$ does not have real rank zero. This is a contradiction because every C^* -subalgebra of $C(\Omega)$ has real rank zero. Thus we conclude that $C(\Omega)$ is a scattered C^* -algebra. \square

Let Ω be a compact Hausdorff space. We remark that if $C(\Omega)$ is scattered, then it has real rank zero. But the converse is false in general. A counterexample is $C(2^\omega)$, which is seen from the proof of Lemma 2.1. On the other hand, from Lemma 2.1 above and this remark, we see that every C^* -subalgebra of $C(\Omega)$ has real rank zero if and only if every C^* -subalgebra of $C(\Omega)$ is scattered.

For a C^* -algebra A , we denote by \hat{A} the spectrum of A which is the set of all equivalence classes of nonzero irreducible representations equipped with the Jacobson topology. As is well known, \hat{A} is locally compact, but it is not necessarily a Hausdorff space. If A is unital, \hat{A} is compact.

Note that every C^* -subalgebra of a scattered C^* -algebra A is scattered ([6, Theorem] or [2, Theorem 1 and Theorem 2]).

Lemma 2.2. *A C^* -algebra A is scattered if and only if every separable abelian C^* -subalgebra of A is scattered.*

Proof. Suppose that every separable abelian C^* -subalgebra of A is scattered. If A is not unital, we consider the C^* -algebra A_1 obtained by adding an identity 1 to A . If A_1 is scattered, so is A . Without loss of generality, thus we may assume that A is unital. A local characterization of a scattered C^* -algebra is that each self-adjoint element h of the C^* -algebra has a countable spectrum $\text{Sp}_A(h)$ of h in the C^* -algebra ([6, Theorem]). Now we use this characterization.

Let h be any self-adjoint element in A . Since the C^* -subalgebra $C^*(h)$ generated by h and 1 is separable and abelian, by assumption $C^*(h)$ is scattered. Since $C^*(h)$ is separable, with [7, Theorem 3.1] and [8, Theorem 2] combined, we see that the spectrum $\widehat{C^*(h)}$ of $C^*(h)$ is countable. Since $\widehat{C^*(h)}$ is homeomorphic to $\text{Sp}_A(h)$, $\text{Sp}_A(h)$ is countable. Hence it follows from [6, Theorem] that A is scattered. \square

Now we are in a position to give the main theorem in this section.

Theorem 2.3. *Let A be a C^* -algebra. Then the following conditions are equivalent.*

- (1) *A is scattered.*
- (2) *Every C^* -subalgebra of A is AF.*
- (3) *Every separable abelian C^* -subalgebra of A is AF.*
- (4) *Every C^* -subalgebra of A has real rank zero.*
- (5) *Every separable abelian C^* -subalgebra of A has real rank zero.*

Proof. (1) \implies (2). If A is scattered, then every C^* -subalgebra of A is scattered. Since a scattered C^* -algebra is AF ([10, Lemma 5.1]), every C^* -subalgebra of A is AF.

(2) \implies (4). It is well known that every AF C^* -algebra of the sense of Bratteli has real rank zero ([5, 3.1]). Every AF C^* -algebra in the our sense also has real rank zero. In fact, let B be an AF C^* -algebra in the our sense. For any self-adjoint element $h \in B$ and any $\varepsilon > 0$, there exists an element a in some finite-dimensional C^* -subalgebra B_0 of B such that $\|h - a\| < \varepsilon$. Then we have $\|h - \frac{a+a^*}{2}\| < \varepsilon$. Hence $\|h - k\| < \varepsilon$ for some self-adjoint element $k \in B_0$. Since B_0 is a finite-dimensional C^* -algebra, k has finite spectrum. Thus the set of all self-adjoint elements with finite spectrum of B is dense in the set of all self-adjoint elements of B . This shows that B has real rank zero ([5, Theorem 2.6]), from which condition (4) follows.

(4) \implies (5). This is obvious.

(5) \implies (1). We assume that every separable abelian C^* -subalgebra of A has real rank zero. Take any separable abelian C^* -subalgebra B of A . Then by assumption, B has real rank zero. First we assume that B has identity. Since every C^* -subalgebra C of B is also a separable abelian C^* -subalgebra of A , C has real rank zero. It hence follows from Lemma 2.1 that B is scattered. Then A is scattered by Lemma 2.2.

Next we assume that B does not have identity. Let B_1 be the C^* -algebra obtained by adjoining an identity 1 to B . Since B has real rank zero, B_1 also does so by the definition of real rank. From the discussion of the unital C^* -algebra case above, we see that B_1 is scattered. Hence B is scattered. Then A is scattered by Lemma 2.2.

(2) \implies (3). This is obvious.

(3) \implies (5). The proof is similar to that of (2) \implies (4). Or this implication easily follows also from [5, 3.1]. \square

3. AMENABILITY

In this section, we consider the question of what kind of C^* -algebra can be characterized by some property all the C^* -subalgebras possess, besides scattered C^* -algebras we have considered in the previous section.

First we briefly review the definition of amenability of a C^* -algebra. Let A be a C^* -algebra. We say that a Banach space X is called a Banach A -module if it is a two-sided A -module and there exists a constant $K > 0$ such that for any $a \in A$ and $x \in X$ we have

$$\|ax\| \leq K\|a\|\|x\| \quad \text{and} \quad \|xa\| \leq K\|x\|\|a\|.$$

For a Banach A -module X , the dual space X^* of X becomes a Banach A -module by

$$(af)(x) = f(xa) \quad \text{and} \quad (fa)(x) = f(ax)$$

for all $a \in A$ and $f \in X^*$. A bounded linear map D from A into X^* is called a derivation if it satisfies $D(ab) = aD(b) + D(a)b$ for $a, b \in A$. If a derivation D from A into X^* is given by $D(a) = af - fa$ for some $f \in X^*$, then D is called inner. A C^* -algebra A is called *amenable* if every bounded derivation from A into X^* is inner for all Banach A -modules X . When A does not have identity, we denote again by A_1 the C^* -algebra obtained from A by adjunction of an identity 1. Then we can make a Banach A -module X into an A_1 -module by $x1 = 1x = x$ for $x \in X$, and D can be extended to a derivation from A_1 into X^* by defining $D(1) = 0$.

A C^* -algebra A is called *strongly amenable* if for every Banach A -module X , every bounded derivation D from A into X^* is given by $D(a) = af - fa$ with some f in the weakly* closed convex hull of $\{-D(u)u^* | u \in U(A_1)\}$, where $U(A_1)$ denotes the unitary group of A_1 . Obviously a strongly amenable C^* -algebra is amenable.

Recall that a C^* -algebra A is said to be *nuclear* if $A \otimes_{\max} B = A \otimes_{\min} B$ for any C^* -algebra B , where \otimes_{\max} and \otimes_{\min} denote the maximal C^* -tensor product and the minimal C^* -tensor product, respectively. Equivalently, a C^* -algebra A is nuclear if the identity map on A is nuclear, that is, for any elements $x_1, x_2, \dots, x_k \in A$ and any $\varepsilon > 0$, there exist a natural number n and completely positive contractions $\Phi : A \rightarrow M_n(\mathbb{C})$ and $\Psi : M_n(\mathbb{C}) \rightarrow A$ such that $\|\Psi \circ \Phi(x_i) - x_i\| < \varepsilon$ for all $i = 1, 2, \dots, k$, where $M_n(\mathbb{C})$ is the $n \times n$ -matrix algebra (e.g., [1, IV.3.1.5]). It is well-known that a C^* -algebra is amenable if and only if it is nuclear ([1, IV.3.3.15]).

In [2, Theorem 1], Blackadar showed that a type I C^* -algebra contains a non-nuclear C^* -subalgebra. This result yields the implication (4) \implies (1) in the Theorem 3.1 below. To show [2, Theorem 1], he constructed an example of a non-nuclear C^* -subalgebra in the type I C^* -algebra. The construction of the example is not easy. In the below, we present an alternative proof of the implication (4) \implies (1) in the Theorem 3.1 based on a well-known theorem of Glimm [11, 6.7.4].

Theorem 3.1. *Let A be a C^* -algebra. Then the following conditions (1) – (5) are equivalent.*

(1) *A is of type I.*

(2) *Every C^* -subalgebra of A is strongly amenable.*

- (3) *Every separable C^* -subalgebra of A is strongly amenable.*
- (4) *Every C^* -subalgebra of A is nuclear.*
- (5) *Every separable C^* -subalgebra of A is nuclear.*

Proof. (1) \implies (2). This follows from the fact that every type I C^* -algebra is strongly amenable ([4, Theorem 7.9]).

(2) \implies (4). This follows from the fact that every strongly amenable C^* -algebra is amenable, hence nuclear.

(4) \implies (1). Assume that A is not of type I. Then there is a C^* -subalgebra B of A such that there exists a surjective homomorphism π from B onto the CAR algebra \mathcal{F} (see [11, 6.7.4]). Let $C_r^*(F_2)$ be the reduced group C^* -algebra of the free group F_2 on two generators. Since $C_r^*(F_2)$ is a separable exact C^* -algebra (cf. [14, 2.5.3]), it follows from [1, IV.3.4.18 (iv)] that there is a C^* -subalgebra C of \mathcal{F} such that there exists a surjective homomorphism ρ from C onto $C_r^*(F_2)$. Since a homomorphic image of a nuclear C^* -algebra is nuclear and since $C_r^*(F_2)$ is not nuclear (cf. [1, II.9.4.6] or [14, 2.4]), C is not a nuclear C^* -subalgebra of \mathcal{F} . Hence $\pi^{-1}(C)$ is not a nuclear C^* -subalgebra of B . Thus we see that A contains the C^* -subalgebra $\pi^{-1}(C)$ which is not nuclear. This is a contradiction.

(2) \implies (3) \implies (5). These are trivial.

(5) \implies (4). Let B be a C^* -subalgebra of A . Take any elements x_1, x_2, \dots, x_k in B and any $\varepsilon > 0$. Let B_0 be the C^* -subalgebra of B generated by x_1, x_2, \dots, x_k . Since B_0 is separable, it follows from Condition (5) that B_0 is nuclear. Hence there exist a natural number n and completely positive contractions $\Phi : B_0 \rightarrow M_n(\mathbb{C})$ and $\Psi : M_n(\mathbb{C}) \rightarrow B_0 \subset B$ such that $\|\Psi \circ \Phi(x_i) - x_i\| < \varepsilon$ for all $i = 1, 2, \dots, k$. By Arveson's Extension Theorem [1, II.6.9.12], there exists a complete positive contraction $\widehat{\Phi} : B \rightarrow M_n(\mathbb{C})$ which extends Φ . Thus we obtain that $\|\widehat{\Phi} \circ \Psi(x_i) - x_i\| < \varepsilon$ for all $i = 1, 2, \dots, k$, which shows that B is nuclear. \square

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